SWITCHING CONTROL OF TIME-VARYING DISCRETE PLANTS

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Abstract

This paper describes a supervised discrete-time switching control algorithm. The supervisor chooses the controller among a finite set of candidate controllers. The process is assumed to be modelled by a linear time-varying discrete system. The process is described by a system that commutes among a finite set of discrete linear time-invariant models. This paper proves the stability of the closed loop system if the time interval between consecutive commuting instants is large enough.

1 Introduction

This paper describes and analyzes a supervised discrete-time switching control algorithm. The supervisor chooses the controller, among a set of candidate controllers, comparing a bank of shared-state estimators and choosing the controller that corresponds to the 'best fitted' model. Such control systems have been proposed in [1, 5] considering linear time-invariant plants. The problem of the robustness with unmodelled dynamics have been studied under this approach in subsequent work [6]. A switching controller for time-varying discrete plants based on hidden Markov models has also been proposed in [4] without a stability analysis.

In this paper the the control scheme follows the one proposed in [5] and sufficient conditions are derived to ensure the stability of the controlled system with a linear time-varying discrete plant.

In section 2 the problem is formulated and the control algorithm is presented. In section 3 the main result is presented and used to prove the stability of the overall system. Simulation results are shown in section 4. Section 5 summarizes some conclusions. Most of the detailed proofs are presented in appendix.

2 Problem Formulation

Consider the set of n admissible linear plants $\{P_1, P_2, ..., P_n\}$ known *a priori* and controllers $\{K_1, K_2, ..., K_n\}$, and admit that process P commute among the elements of $\{P_1, P_2, ..., P_n\}$. Further admit, that the closed loop composed by $\{K_i, P_i\}$ is stable with margin at least λ_{CL} and at each instant k, the active controller $K_{\sigma(k)}$

is decided by a estimator based supervisor with output $\sigma(k)$, Figure 1.

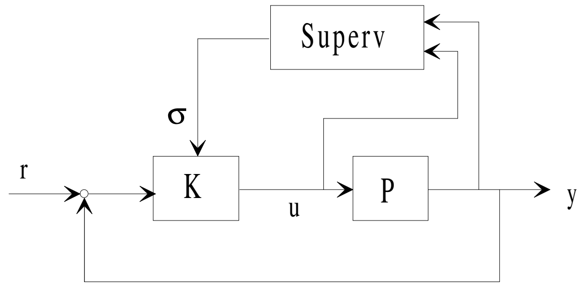

Figure 1: Feedback loop and Supervisor.

Admitting the plant varies slowly enough (with a minimum interval T_{Dp} between changes), and the supervisor is defined as in [1, 8], the problem of interest is to analyze the stability (in the BIBO sense) of the overall system composed by the plant, controller, and supervisor.

Figure 1 depicts a block diagram of the feedback control loop in which an integrator may be included in series with the plant (thus defining a new plant block) to ensure that tracking error goes to zero whenever de reference $r(k)$ and the plant dynamics are constant, as well as smooth transitions when K_σ changes.

2.1 Switching Control

Around each operating point defined by the occurrence of a given operating regime, the plant is modelled by discrete linear time-invariant system P_i, which is assumed to be detectable and stabilizable. A linear local controller K_i is associated to each of the above local models. The mission of the supervisor is to compare the set of admissible plants with the true one and switch among controllers accordingly. Consider a supervisor constituted by a multiple estimator, a performance evaluator and a switching logic scheme, Figure 2. The estimator generate estimates of the system output, the performance evaluator makes an average of the squared estimator errors, and the switching logic scheme commutes among controllers according to the performance signals assuring that each controller remains active at least T_D sampling instants, denoted as the *dwell-time*.

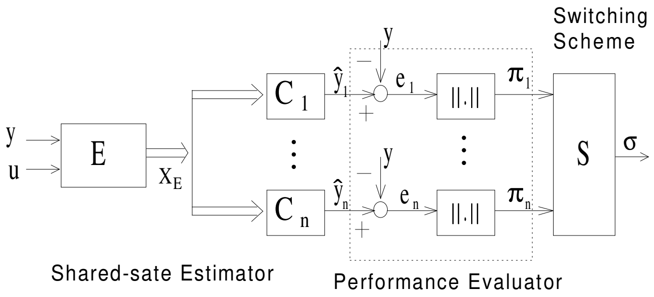

Figure 2: Supervisor constituted by estimator, performance evaluator and a switching logic scheme.

A convenient way to construct the multiple estimator as well as the bank of controllers is in a shared-state form. This form, which is a non-minimal realization, has the advantage of concentrating the dependence of the parameters in the output equation. Thus, define the shared-state multiple estimator

$$x_E(k+1) = \begin{bmatrix} A_E & 0 \\ 0 & A_E \end{bmatrix} x_E(k) +$$

$$\begin{bmatrix} b_E \\ 0 \end{bmatrix} y(k) + \begin{bmatrix} 0 \\ b_E \end{bmatrix} u(k) \tag{1}$$

$$\hat{y}_p(k) = C_p x_E(k) \tag{2}$$

where A_E has eigenvalues inside de circle of radius $\lambda_E < 1$; (A_E, b_E) is a controllable pair and C_p depends on the parameters of the $p-th$ plant model.

Consider the shared-state multiple controller.

$$x_C(k+1) = \begin{bmatrix} A_C & 0 \\ 0 & A_C \end{bmatrix} x_C(k) +$$

$$\begin{bmatrix} b_C \\ 0 \end{bmatrix} (r(k) - y(k)) + \begin{bmatrix} 0 \\ b_C \end{bmatrix} u(k) \tag{3}$$

where A_C as eigenvalues inside the circle of radius $\lambda_C < 1$, and (A_C, b_C) is a controllable pair. The control signal is given by

$$u_\sigma(k) = f_\sigma x_C(k) + g_\sigma(r(k) - y(k)) \tag{4}$$

where f_σ and g_σ are vectors depending on the parameters of the σ-th local controller. Consider the performance evaluator defined by,

$$\pi_i(k+1) = \lambda_\pi \pi_i(k) + e_i^2(k) \qquad (5)$$

and the switching logic scheme defined by the following pseudo code

```
counter:=0
sigma:=1
while 1
begin
  wait_for_sample
  counter:=counter+1
  if counter >= T_D then
      if pi(sigma) > min( pi )
          sigma := argmin( pi )
          counter:=0
      endif
  endif
end
```

This switching logic scheme ensures that the controller corresponding to the minimum performance signal, π_i, is applied if T_D sampling instants have passed since the last controller commutation, otherwise the controller remains unchanged.

The interconnection of the above defined subsystems is analyzed, from the stability point of view, for linear time-invariant plants in [5] and [1] for continuous and discrete-time cases, respectively.

2.2 Composite System

The key idea to proof the stability of the controlled system in the time-invariant case is to build a composite system with state $x = [x'_E(k)\ x'_C(k)]'$, with the i-th estimator error $e_i(k)$ as the input, the control signal $u(k)$, the tracking error $e_T(k)$, and the j-th estimator error $e_j(k)$ as outputs. The stability of the closed loop system $\{K_i, P_i\}$ (for every index i) implies the detectability of the composite system through $e_i(k)$ [2]. According to the exact matching hypothesis there will be (at least) one prediction error which will decay exponentially to zero. Let $\mathcal{P}^*$ be a set of indices known to include only indices j such that e_j decays exponentially to zero. It is proven in [5] the active controller, after a finite but unknown time period, will be within $\mathcal{P}^*$ thus assuring the composite state $[x'_E(k)\ x'_C(k)]'$ goes exponentially to zero. This paper extends this idea for time-varying plants. Note that the preceding argument cannot be used with time-varying plants since the active controller belongs to $\mathcal{P}^*$ after an unknown time period and every time the plant changes the set $\mathcal{P}^*$ may also change.

From (1-4) through simple transformations similar to those presented in[5] it is possible to build the composite system with inputs $r(k)$, $e_i(k)$, and with state $x(k) = [x'_E(k)\ x'_C(k)]'$,

$$\begin{aligned} x(k+1) &= A_{ij}x(k) + d_i\, e_j(k) + d'_j r(k) && (6)\\ u(k) &= f_{ij}x(k) + g_i\, e_j(k) + g'_j r(k) && (7)\\ y(k) &= c_j x(k) - e_j(k) + r(k) && (8)\\ e_p(k) &= c_{pj}x(k) + e_j(k) && (9) \end{aligned}$$

where matrix A_{ij} is such that $A_{\sigma\sigma}$ is stable with all eigenvalues inside the circle of radius [1, 5]. Considering the process between commutations defined by

$$\begin{aligned} x_P(k+1) &= A_P x_P(k) + B_P\, u(k) && (10)\\ y(k) &= C_P x_P(k) && (11) \end{aligned}$$

which is assumed detectable and stabilizable with unobservable or uncontrollable modes with eigenvalues inside the circle of radius $\lambda_\omega < 1$. Then, by definition, there exists K such that $A_P + K\, C_P$ as all eigenvalues inside the circle of radius λ_ω. Thus, from equation (10), the following relation is yielded

$$x_P(k+1) = (A_P + K C_P)x_P(k) + B_P u - K y \quad (12)$$

From (5), (6-9) and (12) it is possible to describe the relations between $[x'_E\, x'_C]'$, $x_P(k)$ and $\pi(k)$ by the block diagram in figure 3.

The main idea to verify the system stability is to devise a relation between the estimator error of the correct model, e_{p^*}, and e_i. If e_{p^*} is bounded and the dynamic matrix A_{ij} is stable, since $A_P + KC_P$ and λ_π are stable, then the overall system will be stable (for every time period with constant plant). Further, if the overall system is exponentially uniformly stable then there will be a dwell-time period for the plant such that the overall systems with time-varying plant will be stable. It should be noted that the system described in figure 3 is parameter-dependent. Setting indices i and j to be the

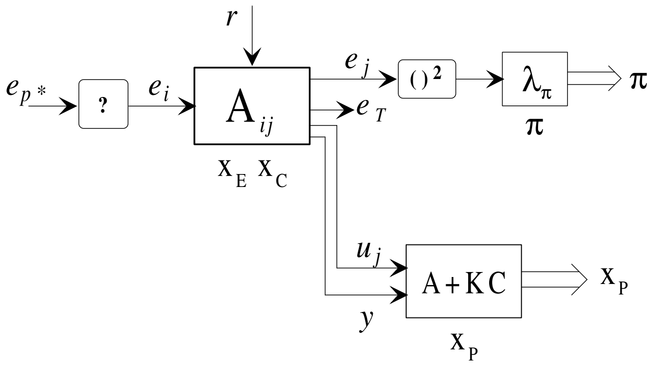

Figure 3: Block diagram describing the relations between $x_E(k)$, $x_C(k)$, $x_P(k)$ and $\pi(k)$.

active controller index, $i = j = \sigma(k)$, yields a stable matrix $A_{\sigma\sigma}$ for the composite system $[x'_E\, x'_C]'$, with eigenvalues inside the circle of radius $\lambda_{CECL} = max(\lambda_C, \lambda_E, \lambda_{CL})$ [1, 5]. However there is no obvious way to bound e_σ as a function of e_{p^*} since the estimator error signal e_σ can be occasionally much larger then e_{p^*}.

3 Main result

The following assumptions are made.

Assumption 1 *The plant dynamic behavior associated to index p^* varies at time instants $k_1^*, k_2^*, k_3^*, \ldots$, so the process dynamic is kept constant in the intervals $k_i^* \leq k < k_{i+1}^*$.*

Assumption 2 *The controller associated to index σ varies at time instants $k_1, k_2, k_3, \ldots$, so the controller is is kept constant in the intervals $k_i \leq k < k_{i+1}$.*

Assumption 3 - Exact Matching *Within an interval where the process is kept constant there is always a model (with index $p^*(k)$) that exactly matches the process. This implies that there is (at least) one estimator error $e_{p^*}(k)$ that goes exponentially to zero.*

Further, as the process uncontrollable modes are inside the circle λ_ω and the estimator ones are inside the circle λ_E, it is easily seen that the estimator error corresponding to the correct model is bound by

$$|e_{p^*}(k)| \leq \gamma \lambda_{\omega E}^{k-k_0^*} |[x_E(k_0^*)\, x_P(k_0^*)]| \qquad (13)$$

where k and k_0^* belong to the same time interval $k_i^* \leq k < k_{i+1}^*$, and $\lambda_{\omega E} = max(\lambda_\omega, \lambda_E)$.

For time-varying plants, a key step to the stability proof of of the overall system is the definition of the indices i and j of the composite system. Let $j(k) = \sigma(k)$ and $i(k) = \bar{\sigma}(k)$ where $\bar{\sigma}$ is defined as follows. For each interval $k_i \leq k < k_{i+1}$ let

$$\overline{\sigma}(k) = \begin{cases} \sigma(k) & if \quad \pi_{\sigma(k_i)}(k_i) \leq \pi_{p^*}(k_b^*)\, \lambda^{k_i - k_b^*} \\ p^* & if \quad \pi_{\sigma(k_i)}(k_i) > \pi_{p^*}(k_b^*)\, \lambda^{k_i - k_b^*} \end{cases} \qquad (14)$$

and for the last interval $k_i \leq k < k_{l+1}^*$ within $[k_l^*, k_{l+1}^* - 1]$, define

$$\overline{\sigma}(k) = \begin{cases} \sigma(k) & if \quad k_b^* - k_u \geq T_D \\ p^* & if \quad k_b^* - k_u < T_D \end{cases} \qquad (15)$$

This definition is intuited by the following lemma.

Lemma 1 *For any two indices i and j and according to the performance signals definition (5)*

$$\pi_i(k_a^*) \geq \pi_j(k_b^*)\, \lambda_\pi^{k_a^* - k_b^*} \implies \pi_i(k) \geq \pi_j(k) \qquad (16)$$

for any k, such that $k_l^* \leq k < k_{l+1}^*$

The proof of Lemma 1 is presented in Appendix. From Lemma 1 it can be seen that function $\pi_{p^*}(k_b^*)\, \lambda_\pi^{k-k_b^*}$ behaves like a "temporary rejection" frontier line. That is, whenever a performance signal goes above that line it will never go below it as long as the process is kept constant.

Thus the definition of $\overline{\sigma}$ is such that $\overline{\sigma} = p^*$ in a certain interval $k_i \leq k < k_{i+1}$ if model with index $\sigma(i)$ has been "temporarily rejected", otherwise $\overline{\sigma} = \sigma$. It is now clear the purpose of the definition of $\bar{\sigma}$. As the number of rejections is not larger than the number of models, which is finite, then the number of time intervals in which $\overline{\sigma} \neq \sigma$ is limited.

From this fact it is proven that the system (which is time-varying) with state matrix $A_{\overline{\sigma}\sigma}$ is stable, and the signal $|e_{\overline{\sigma}}|$ goes exponentially to zero. This is stated in the following key lemmas proven in Appendix.

Lemma 2 *Consider the function $\psi(k)$ such as $\psi(k) = 1$ if $\sigma = p^*$ and $\psi(k) = 0$ otherwise. Then*

$$\sum_{j=k_i^*}^{k_{i+1}^*-1} \psi(k) \leq T_D\,(n-1) \tag{17}$$

where T_D is the controller dwell-time imposed by the switching logic scheme.

Lemma 2 states that the period of time such that $\overline{\sigma} \neq \sigma$, within an interval $k_l^* \leq k < k_{l+1}^*$ is limited.

Lemma 3 *Consider $\bar{\sigma}$ defined by (14-15) then the composite system with state matrix $A_{\bar{\sigma}\sigma}$ is uniformly exponentially stable with stability margin*[1] *$\lambda_{CECL} = max(\lambda_C, \lambda_E, \lambda_{CL})$ within any interval $k_l^* \leq k < k_{l+1}^*$*

Lemma 4 *Consider the switching signal σ according to the dwell-time switching scheme, and $\bar{\sigma}$ defined by (14-15) then within any interval $k_l^* \leq k < k_{l+1}^*$* $e_{\overline{\sigma}(k)}(k)$ goes exponentially to zero

$$|e_{\overline{\sigma}(k)}(k)| \leq \lambda_{\pi\omega E}^{\frac{1}{2}(k-k_0^*)} \times$$

$$\{\, \gamma_{\sigma 1}\|\pi^{\frac{1}{2}}(k_0^*)\| + \gamma_{\sigma 2}\|x_E(k_0^*)\| + \gamma_{\sigma 3}\|x_P(k_0^*)\|\} \tag{18}$$

where $\lambda_{\pi\omega E}^{\frac{1}{2}} = max(\lambda_{\pi}^{\frac{1}{2}}, \lambda_\omega, \lambda_E)$

Thus, from the above lemmas, the composite system is uniformly exponentially stable and depends on the initial conditions of $x_P(k_0^*)$, $\pi(k_0^*)$, $x_E(k_0^*)$ and $x_C(k_0^*)$, as x_P depends linearly from the composite system the same conclusion holds to the process. Simple calculations allow us to bound $\|\pi\|^{1/2}$ as a function of the initial conditions of the overall system at instant k_0^*, and as a function of the time period $k - k_o^*$. This facts are stated in the following theorem whose detailed proof is omitted in this paper.

Theorem 1 *Consider the overall system within an interval such that the process dynamics is constant, defined by equations (5), (6-9), (12), the switching signal σ according to the dwell-time switching scheme, and $\bar{\sigma}$ defined by (14-15), then the overall system with state $[x'_C\, x'_E]'$, x_P, and π is uniformly exponentially stable with stability margin*

$$\lambda_{\pi\omega CECL} = max(\lambda_\pi^{1/2}, \lambda_\omega, \lambda_C, \lambda_E, \lambda_{CL})$$

If a dwell-time condition is imposed to a set uniformly exponentially stable systems then the commutation among those systems will be exponentially stable. This conclusion leads to Theorem 2. The proof of Theorem 2 follows the proof of Lemma 4 in [5], extended to the case when the system commutes among time-varying subsystems.

Theorem 2 *Consider the overall system defined as in Theorem 1, if the process commutes among the admissible models assuring a minimum time between commutations T_{Dp}, then the overall time-varying system is exponentially stable.*

It should be noted that the only situation assuring the tracking error goes to zero is when the reference is zero, that is, the regulation case.

4 Simulation Results

In this section some simulation results are presented to show stability and instability cases of a time-varying switching controlled plant depending on the process dwell-time.

The plant considered in this simulation example is a double cart coupled by a link with a varying stiffness parameter [7], in figure 4. The control problem is to position cart n. 2 applying a force at cart n. 1.

[1] A linear systems with state x is exponentially uniformly stable with stability margin λ, if there exist a finite γ and $0 \leq \lambda < 1$ such that $\|x(k)\| \leq \gamma\lambda^{k-k_0}\|x(k_0)\|$

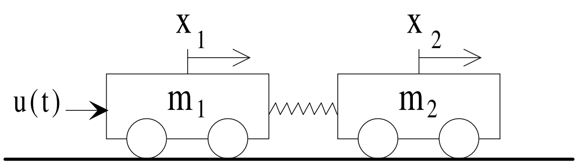

Figure 4: Plant - coupled double cart with variable stiffness.

The controllers and models used in the following simulations are those defined in [7], which were designed according to a weighted H_∞ mixed-sensitivity method [3]. The discretization of the models $\frac{B_i}{A_i}$ and the controllers $\frac{S_i}{R_i}$ was done assuming an ideal zero order hold with sampling time $T_s = 0.1\,s$. Model 1 correspond to a stiffness $\gamma_1 = 0.3$ and model 2 corresponds to $\gamma_2 = 1.0$. In the figures below it can be seen the simulation results of the supervised controlled system, assuming $\lambda_\pi = 0.99$, a dwell-time of $T_D = 10$, and the plant commute between 1 and 2 every 800 sampling periods. Excitation dither noise was added to the manipulated variable with power $\eta^2 = 10^{-9}$. The maximum modulus of the closed loop eigenvalues λ_{km}, where k and m are the indices associated to controller and model respectively, are

$$\lambda_{11} = 0.985 \quad \lambda_{22} = 0.951 \quad \lambda_{12} = 1.025 \quad \lambda_{21} = 1.017$$

Thus the closed loop of each model with the "wrong" controller is unstable.

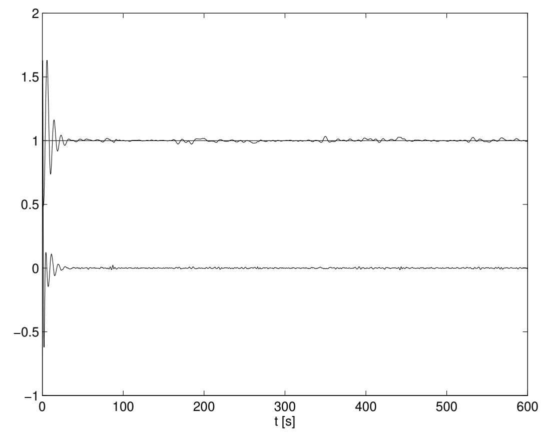

Figure 5: Reference $r = 1$, manipulated variable u, and plant output y.

In figure 5 it can be seen the control signal $u(k)$, the reference $r(k)$ with a constant value $r = 1$, and the process output $y(k)$. It should be noted that since there is an integrator which is intrinsic to process, the equilibrium control signal is $u_{eq} = 0$. So, the process commuting among the admissible models does not influence u_{eq}, this justifies the absence of visible transient responses after each commutation. The index

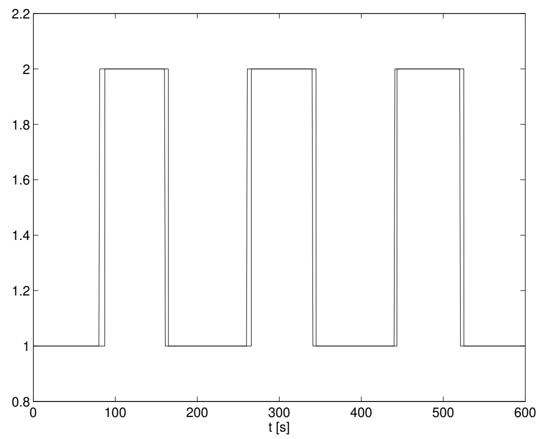

Figure 6: Index of the active process model and supervisor decision, $\sigma(k)$

of the active process and the supervisor decision $\sigma(k)$ are shown in figure 6. The delay between the change in the process and the supervisor decision remains limited between 3 and 6 seconds.

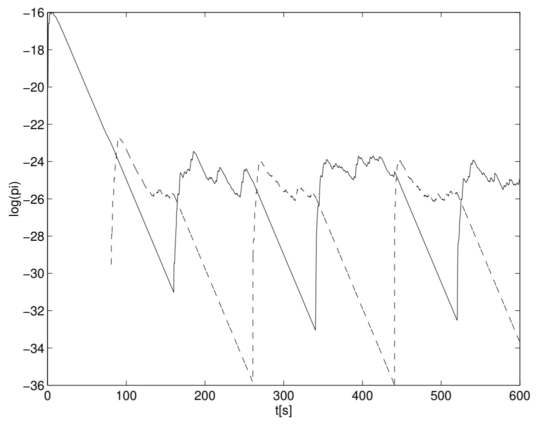

Figure 7: Performance signals logarithm $log(\pi_i)$

Figure 7 represents the performance signals logarithm. It can be seen that at each instant the predictive model that corresponds to the right model decays exponentially to zero (linear decreasing in the logarithmic scale).

A phase diagram of u and y is observed in figure 8. This diagram may be interpreted as the evolution of a closed loop system partial state, which in this particular case goes to un equilibrium point($y_{eq} = 1$, $u_{eq} = 0$).

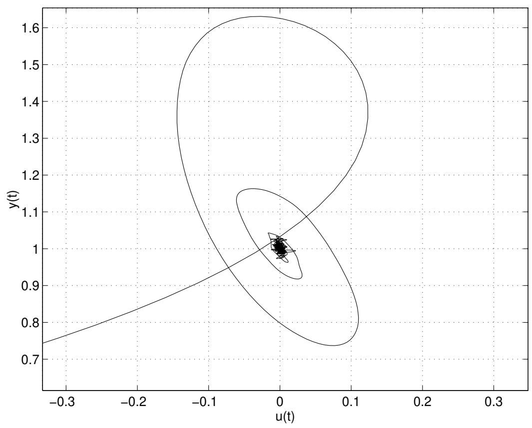

Figure 8: Phase diagram of the manipulated variable u and process output y

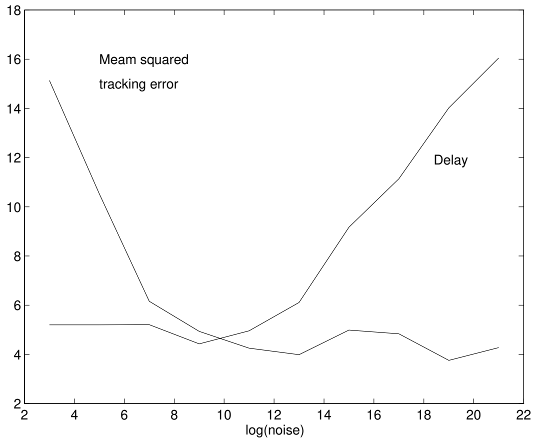

Figure 9: Relation between dither noise ($10^{-3} \leq \eta^2 \leq 10^{-21}$), the supervisor delay Δ_σ and the tracking error e_T. The signals represented are $log(\Delta_\sigma) + 11$ and $0.005\, \bar{e^2}_T$.

According to theoretic results about supervised switching control systems the injection of dither noise is not necessary to ensure closed loop system stability. However, dither noise may influence the performance of the closed loop system. Figure 9 present the effect of dither noise (when the power varies from 10^{-3} to 10^{-21}) onto the delay of the supervisor Δ_σ and the tracking error e_T, assuming the other parameters were kept constant, whit respect to the previous simulations. The supervisor delay is evaluated through a simulation by the sum of all sampling instants in which the model index is different from the supervisor output σ, while the tracking error is evaluated by the mean squared tracking error. To make possible to represent both signals in the same diagram the tracking error is represented through a logarithmic transformation - $log(\bar{e^2}_T) + 11$ - and the delay is multiplies by a factor - $0.005\, \Delta_\sigma$. It can be seen in figure 9 that the dither noise injection reduces de supervisor delay, improving the process identifiability as the dither power is increased till 10^{-8}. On the other hand, in this simulation example, a higher delay in the supervisor does not seem to influence the tracking error that grows for a dither power higher than 10^{-12}. Thus there is an intermedius range for the dither power (between 10^{-8} and 10^{-12}) which generates simultaneously a good process identifiability as well as a small tracking error.

In the following the simulation results are obtained considering a process commuting between model 1 and model 2 in cycles of $26\, s$, figure 10.

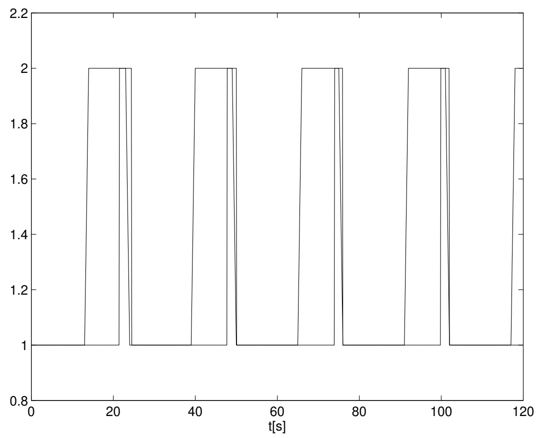

Figure 10: Index of the active process model and supervisor decision σ.

It can be seen, in figures 11 and 12, that the closed loop system has an unstable behavior. The performance signals in figure 12 show that in the period that they correspond to correct model they have a exponential decaying behavior, as had already been observed in previous simulations. Thus the instability is due to the fact that the time period in which it decays to zero is unsufficient to compensate the growing that occurs every time the process commutes. This fact shows in a very intuitively way the necessity to impose a condition on the process change rate to ensure the controlled system stability.

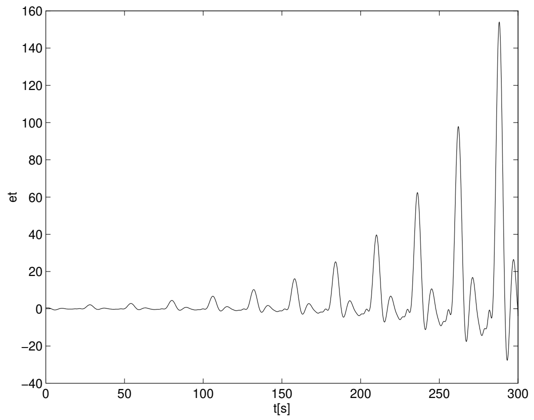

Figure 11: Tracking error e_T.

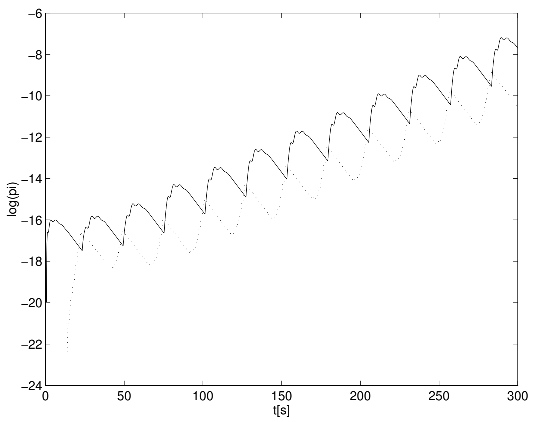

Figure 12: Performance signals ligarithm $log(\pi)$.

5 Conclusions

A switching control algorithm applied to linear time-varying plants as been analyzed. The stability proof has been outlined and conditions have been posed to ensure stability of the controlled system. This condition has been defined as a "dwell-time" on the plant dynamics. Simulation examples have been presented showing the effect of the plant dwell-time in the stability of the controlled system.

Acknowledgements

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Appendix

Proof of Lemma 1 By definition of performance signals

$$\pi_i(k+1) = \lambda_\pi \pi_i(k) + e^2(k) \tag{19}$$

then for $\pi(k)$, and $k > k_0$ yields

$$\pi_i(k) = \lambda_\pi^{k-k_0} \pi_i(k_0) + \sum_{j=k_0}^{k-1} \lambda_\pi^{k-j-1} e_i^2(j) \tag{20}$$

and from

$$\sum_{j=k_0}^{k-1} \lambda_\pi^{k-j-1} e_i^2(j) \geq 0 \tag{21}$$

$$\pi_i(k) \geq \lambda_\pi^{k-k_0} \pi_i(k_0) \tag{22}$$

$$\lambda_\pi^{k_0-k} \pi_i(k) \geq \pi_i(k_0) \tag{23}$$

substituting (23) in the l.h.s. of the first inequality, from (16),

$$\pi_i(k)\, \lambda_\pi^{k_a^*-k} \geq \pi_i(k_a^*) \geq \pi_j(k_b^*)\, \lambda_\pi^{k_a^*-k_b^*} \tag{24}$$

$$\pi_i(k) \geq \pi_i(k_a^*)\, \lambda_\pi^{k-k_a^*} \geq \pi_j(k_b^*)\, \lambda_\pi^{k_a^*-k_b^*+k-k_a^*} \tag{25}$$

$$\pi_i(k) \geq \pi_j(k_b^*)\, \lambda_\pi^{k-k_b^*} \tag{26}$$

Substituting (23) in the r.h.s of (26) for $k < k_b^*$ follows that

$$\pi_i(k) \geq \pi_j(k_b^*)\, \lambda_\pi^{k-k_b^*} \geq \pi_j(k) \tag{27}$$

so, finally

$$\pi_i(k) \geq \pi_j(k) \tag{28}$$

Proof of Lemma 2 Let $\mathcal{C}_i$ be the set of sampling instants k such that $\sigma(k) = i$; let $\phi_i(k)$ be

$$\phi_i(k) = \begin{cases} 0 & se \quad \sigma(k) = \overline{\sigma}(k) \\ 1 & se \qquad c.c. \end{cases} \tag{29}$$

if $k \in \mathcal{C}_i$, and otherwise $\phi_i(k) = 0$, and let k_{ui} be the last switching instant before k, and $\mathcal{I}_{ui}$ the interval $[k_{ui}, k]$. From the definitions of ψ and ϕ_i the following easily derived facts hold.

$$\phi_{p^*}(k) = 0 \tag{30}$$

$$\phi_i(k) = 0 \;\; se \;\; k \leq k_{iu} \tag{31}$$

$$\phi_i(k) = 0 \;\; se \;\; k > k_{iu} \;\; e \;\; k_{iu+1} - k_{iu} > T_D \tag{32}$$

$$\phi_i(k) \leq 1 \;\; se \;\; k > k_{iu} \;\; e \;\; k_{iu+1} - k_{iu} \leq T_D \tag{33}$$

$$\psi(k) = \sum_{i=1}^{n} \phi_i(k) \tag{34}$$

Thus,

$$\sum_{k\in\mathcal{I}} \psi(k) = \sum_{i=1}^{n} \sum_{k\in\mathcal{C}_i} \psi(k) \tag{35}$$

$$\sum_{k\in\mathcal{I}} \psi(k) = \sum_{i=1}^{n} \sum_{k\in\mathcal{C}_i} \phi_i(k) \tag{36}$$

$$\sum_{k\in\mathcal{I}} \psi(k) = \sum_{i=1}^{n} \left(\sum_{k\in\mathcal{C}_i \,\wedge\, k\notin\mathcal{I}_{iu}} \phi_i(k) + \sum_{k\in\mathcal{I}_{iu}} \phi_i(k) \right) \tag{37}$$

according to (31),

$$\sum_{k\in\mathcal{C}_i \wedge k\notin\mathcal{I}_{iu}} \phi_i(k) = 0 \tag{38}$$

$$\sum_{k\in\mathcal{I}} \psi(k) = \sum_{i=1}^{n} \sum_{k\in\mathcal{I}_{iu}} \phi_i(k) \tag{39}$$

$$\sum_{k\in\mathcal{I}} \psi(k) = \sum_{1\leq i\leq n \,\wedge i\neq p^*} \sum_{k\in\mathcal{I}_{iu}} \phi_i(k) + \sum_{k\in\mathcal{I}_{p^*u}} \phi_p^*(k) \tag{40}$$

from (30), yields

$$\sum_{k\in\mathcal{I}} \psi(k) = \sum_{1\leq i\leq n \,\wedge i\neq p^*} \sum_{k\in\mathcal{I}_{iu}} \phi_i(k) \tag{41}$$

depending on $\mathcal{I}_{iu}$, from (32) and (33)

$$\sum_{k\in\mathcal{I}_{iu}} \phi_i(k) = 0 \quad for \quad k_{iu+1} - k_i u > T_D \tag{42}$$

$$\sum_{k\in\mathcal{I}_{iu}} \phi_i(k) = \sum_{k\in\mathcal{I}_{iu}} 1 \quad for \quad k_{iu+1} - k_i u \leq T_D \tag{43}$$

$$\sum_{k\in\mathcal{I}_{iu}} \phi_i(k) \leq T_D \quad for \quad k_{iu+1} - k_i u \leq T_D \tag{44}$$

$$\sum_{k\in\mathcal{I}} \psi(k) \leq \sum_{1\leq i\leq n \,\wedge i\neq p^*} T_D \leq T_D \sum_{1\leq i\leq n \,\wedge i\neq p^*} 1 \tag{45}$$

the number of indices is n, under the perfect matching assumption $1 \leq p^* \leq n$, so the sum is done on $n-1$ terms, which gives

$$\sum_{k\in\mathcal{I}} \psi(k) \leq T_D(n-1) \tag{46}$$

Proof of Lemma 3

$$A_{\overline{\sigma}\sigma} = \psi A_{\overline{\sigma}\sigma} + (1-\psi) A_{\overline{\sigma}\sigma} \tag{47}$$

according to the definition of ψ

$$A_{\overline{\sigma}\sigma} = \psi A_{p^*\sigma} + (1-\psi) A_{\sigma\sigma} \tag{48}$$

$$A_{\overline{\sigma}\sigma} = A_{\sigma\sigma} + \psi(A_{p^*\sigma} - A_{\sigma\sigma}) \tag{49}$$

Considering that,

$$\sum_{k=k_0^*}^{\infty} \|\psi(A_{p^*\sigma} - A_{\sigma\sigma})\| \leq \sum_{k=k_0^*}^{\infty} |\psi| \, \|(A_{p^*\sigma} - A_{\sigma\sigma})\| \tag{50}$$

$$\sum_{k=k_0^*}^{\infty} \|\psi(A_{p^*\sigma} - A_{\sigma\sigma})\| \leq \alpha \sum_{k=k_0^*}^{\infty} |\psi| \tag{51}$$

applying Lemma 3, yelds

$$\sum_{k=k_0^*}^{\infty} \|\psi(A_{p^*\sigma} - A_{\sigma\sigma})\| \leq \alpha\,(n-1)T_D \qquad (52)$$

where $\alpha = max\|(A_{p^*\sigma} - A_{\sigma\sigma})\|$. As the system defined by $A_{\sigma\sigma}$ is uniformly stable, applying a standard perturbation result and the Bellman-Gronwall theorem [9], the system defined by matrix $A_{\bar{\sigma}\sigma}$ is uniformly stable with the same exponential decaying.

Proof of Lemma 4

Consider the interval $[\,k_a^*\; k_b^*]$ From the performance signals definition

$$\pi_{\overline{\sigma}(k_b^*)}(k_b^*) \leq \pi_{p^*}(k_b^*) \qquad (53)$$

$$\pi_{\overline{\sigma}(k_b^*)}(k_b^*) \leq \pi_{p^*}(k_a^*)\lambda_\pi^{k_b^*-k_a^*} + \sum_{j=k_a^*}^{k_b^*-1} \lambda_\pi^{k_b^*-j-1}\, e_{p^*}^2(j) \qquad (54)$$

from (13)

$$|e_{p^*}(k)| \leq \lambda_{\omega E}^{k-k_a^*}\, \|[x'_E(k_a^*), x'_P(k_a^*)]'\|\, C \qquad (55)$$

$$|e_{p^*}(k)|^2 \leq (\lambda_{\omega E}^2)^{k-k_a^*}\, \|[x'_E(k_a^*), x'_P(k_a^*)]'\|^2\, C^2 \qquad (56)$$

From the above and Lemma 3, the composite system is a exponentially stable system with an exponentially decaying input e_{p^*}. Thus,

$$\pi_{\overline{\sigma}(k_b^*)}(k_b^*) \leq \lambda_\pi^{k_b^*-k_a^*}\pi_{p^*}(k_a^*)$$

$$+ \lambda_{\pi\omega E}^{k_b^*-k_a^*} C\|[x'_E(k_a^*), x'_P(k_a^*)]'\|^2 \qquad (57)$$

where $\lambda_{\pi\omega E} > max(\lambda_\pi, \lambda_\omega^2, \lambda_E^2)$ By definition $|e_i(k)|^2 \leq \pi_i(k)$, so

$$|e_{\overline{\sigma}(k_b^*)}(k_b^*)|^2 \leq \pi_{\overline{\sigma}(k_b^*)}(k_b^*) \leq \lambda_\pi^{k_b^*-k_a^*}\pi_{p^*}(k_a^*)$$

$$+ \lambda_{\pi\omega E}^{k_b^*-k_a^*} C\|[x'_E(k_a^*), x'_P(k_a^*)]'\|^2 \qquad (58)$$

redefining the constants it yields Lemma 4.

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